

# Magic baseline and magic energy in neutrino oscillation with non-standard interactions

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## Abstract

We have discussed conditions under which probability of oscillation ( $\nu_e \rightarrow \nu_\mu$ ) is independent of CP violating phase  $\delta$ . The condition of magic baseline on its length is well-known. We have proposed another condition which is on neutrino energy. We have shown that magic baseline condition is not possible in general, for small  $\theta_{13}$  with non-standard interaction and for large  $\theta_{13}$  with both standard and non-standard interactions. However, neutrino energy condition is possible for such cases as well as for cases where magic baseline condition is applicable. We have discussed how one may resolve hierarchy problem for neutrino masses by using such energy condition. For a baseline of length 650 Km, using this energy condition we discuss the possible number of  $\mu^-$  events at the detector for a period of 5 years and also the sensitivity in measurement of  $\cos^2 \theta_{13}$ .

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The probability of oscillation of different flavors of neutrinos depends on various parameters present in the neutrino mixing matrix - the PMNS matrix [1] as well as the mass squared differences. Although two angles  $\theta_{12}$  and  $\theta_{23}$  are known with some accuracy but there is only upper bound for  $\theta_{13}$  [2] and CP violating phase  $\delta$  is unknown. Although mass squared differences for different neutrinos are known but the sign of  $\Delta m_{31}^2$  (where  $\Delta m_{ij}^2 = m_i^2 - m_j^2$  and  $m_i$  is the mass of  $i$ -th neutrino) is unknown. Due to correlations among these unknowns there are ambiguities [3] in analysing neutrino oscillation data. To reduce such ambiguities it is useful to choose suitable baseline [4]. Particularly, magic baseline [5, 6, 7, 8] is useful for some specific length for which the perturbative expression of probability  $P_{\nu_e \rightarrow \nu_\mu}$  is independent of  $\delta$  upto order  $\alpha^2$  (where  $\alpha = \Delta m_{21}^2 / \Delta m_{31}^2$ ). Then it is easier to find out other parameters apart from  $\delta$  which plays the role in neutrino oscillation. However, using magic baseline has its limitations also - namely (a) it may not be always possible to place the detector at a magic baseline distance from the source of neutrino production, (b) if there is non-standard interaction (NSI) [9] then we have shown that it is difficult to get  $\delta$  independent probability  $P_{\nu_e \rightarrow \nu_\mu}$  using the magic baseline condition.

In this work we present another condition - which may be termed as magic energy condition under which also the probability of oscillation will be independent of  $\delta$  upto order  $\alpha^2$ . Considering this condition one might be able to circumvent the above two shortcomings of the magic baseline. Unlike magic baseline condition this condition depends on length of the baseline also apart from its dependence on  $\sqrt{2}G_F n_e$  (Here  $n_e$  is the electron number density of the matter). To use the magic energy condition, one option could be to analyse oscillation data in small energy bins. However, the other better option could be to use monoenergetic neutrino beam at the source as proposed in recent years [10, 11]. The idea is about using nucleus which absorbs an electron and emits a neutrino. By accelerating the mother nuclei with suitable Lorentz boost factor one may get the suitable neutrino energy which is satisfied by magic energy condition. Due to the monoenergetic nature of the beam it is expected to have better precision in finding various neutrino oscillation parameters.

The magic baseline condition was initially obtained using the perturbative expansion for small  $\theta_{13}$  with Standard Model interaction (SMI) [5]. Here, we obtain the modified form of magic baseline condition for both small and large  $\theta_{13}$  and for both SMI and NSI. Besides, we obtain the magic neutrino energy condition for both SMI and NSI and also for both small and large  $\theta_{13}$  as allowed by present experiment [2]. Finally, we have compared advantages and disadvantages in considering magic baseline condition and magic energy condition in experiments.

Flavor eigenstates  $\nu_\alpha$  may be related to mass eigenstates of neutrinos  $\nu_i$  as

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle, \quad U = R_{23} R_{13}(\delta) R_{12} \quad \text{and} \quad i = 1, 2, 3, \quad (1)$$

where  $U$  is PMNS matrix [1] and  $R_{ij}$  are the rotation matrices. General probability expression for oscillation of neutrino of flavor  $l$  to neutrino flavor  $m$  in matter (satisfying adiabatic condition for the density of matter) is given by

$$P(\nu_l \rightarrow \nu_m) = \delta_{lm} - 4 \sum_{i>j} \operatorname{Re}[J_{ij}^{lm}] \sin^2 \Delta'_{ij} + 2 \sum_{i>j} \operatorname{Im}[J_{ij}^{lm}] \sin 2\Delta'_{ij} \quad (2)$$

where  $J_{ij}^{lm} = U'_{li} U'_{lj}^* U'_{mi}^* U'_{mj}$  and  $\Delta'_{ij} = \Delta' m_{ij}^2 L / (4E)$ . Here  $\Delta' m_{ij}^2 = m_i'^2 - m_j'^2$  and label (') indicates the neutrino matter interaction induced quantities corresponding to those quantities in vacuum. Let us write  $x = \Delta'_{31}$ ,  $y = \Delta'_{32}$  and  $z = \Delta'_{12}$ . Using trigonometric identities :  $-\sin^2 x + \sin^2 y - \sin^2 z = 2 \sin x \cos y \sin z$  and  $-\sin 2x + \sin 2y - \sin 2z = -4 \sin x \sin y \sin z$  where  $x$ ,  $y$ ,  $z$  obey the relationship  $x = y - z$ ,  $y = x + z$  and  $z = y - x$  and putting the condition  $\sin z = 0$ , the probability expression  $P_{\nu_e \rightarrow \nu_\mu}$  can be written as

$$\begin{aligned} P_{\nu_e \rightarrow \nu_\mu} = & -4 \left( \operatorname{Re} \left[ U'_{13} U'_{11}^* U'_{23}^* U'_{21} \right] + \operatorname{Re} \left[ U'_{13} U'_{12}^* U'_{23}^* U'_{22} \right] \right) \sin^2 \Delta'_{31} \\ & -2 \left( \operatorname{Im} \left[ U'_{13} U'_{11}^* U'_{23}^* U'_{21} \right] + \operatorname{Im} \left[ U'_{13} U'_{12}^* U'_{23}^* U'_{22} \right] \right) \sin 2\Delta'_{31}. \end{aligned} \quad (3)$$

The condition  $\sin z = 0$  implies

$$|\Delta'_{21}| = \pm n\pi \quad \text{where } n \text{ is positive integer.} \quad (4)$$

Similarly, under the condition  $\sin x = 0$ , the probability expression  $P_{\nu_e \rightarrow \nu_\mu}$  can be written as

$$\begin{aligned} P_{\nu_e \rightarrow \nu_\mu} = & -4 \left( \operatorname{Re} \left[ U'_{12} U'_{11}^* U'_{22}^* U'_{21} \right] + \operatorname{Re} \left[ U'_{13} U'_{12}^* U'_{23}^* U'_{22} \right] \right) \sin^2 \Delta'_{12} \\ & -2 \left( \operatorname{Im} \left[ U'_{12} U'_{11}^* U'_{22}^* U'_{21} \right] - \operatorname{Im} \left[ U'_{13} U'_{12}^* U'_{23}^* U'_{22} \right] \right) \sin 2\Delta'_{12} \end{aligned} \quad (5)$$

and here the condition  $\sin x = 0$  implies

$$|\Delta'_{31}| = \pm n\pi \quad \text{where } n \text{ is positive integer.} \quad (6)$$

The condition in eq.(4) or in (6) essentially either fixes the length of baseline or the energy of the neutrino beam. To determine elements of  $U'$  and  $\Delta'_{ij}$  we shall use the perturbative approach for small and large  $\theta_{13}$  separately.

We discuss in brief the perturbative approach here. The diagonal neutrino mass matrix is approximately given by

$$m \approx \Delta m_{31}^2 \operatorname{diag}(0, \alpha, 1). \quad (7)$$

The effective Hamiltonian induced by interaction of matter with neutrinos is written in weak interaction basis as

$$H \approx \frac{\Delta m_{31}^2}{2E} R_{23} M R_{23}^\dagger \quad (8)$$

where

$$M = R_{13} R_{12} \frac{m}{\Delta m_{31}^2} R_{12}^\dagger R_{13}^\dagger + \text{diag}(A, 0, 0) + R_{23}^\dagger \begin{pmatrix} 0 & X & Y \\ X^* & B & C \\ Y^* & C^* & D \end{pmatrix} R_{23}. \quad (9)$$

In equation (9)

$$A = \frac{2E\sqrt{2}G_F n_e}{\Delta m_{31}^2}, \quad X = \frac{2E\epsilon_{12}}{\Delta m_{31}^2}, \quad Y = \frac{2E\epsilon_{13}}{\Delta m_{31}^2}, \quad B = \frac{2E\epsilon_{22}}{\Delta m_{31}^2}, \quad C = \frac{2E\epsilon_{23}}{\Delta m_{31}^2}, \quad D = \frac{2E\epsilon_{33}}{\Delta m_{31}^2}, \quad (10)$$

where  $A$  is considered due to Standard model interaction of neutrinos with electron.  $\epsilon_{12}$ ,  $\epsilon_{13}$ ,  $\epsilon_{22}$ ,  $\epsilon_{23}$  and  $\epsilon_{33}$  are considered due to NSI of neutrinos with matter [9] (e.g, in  $R$  violating Supersymmetric Models neutrinos may interact with down type quarks through squark exchange and may interact with electron through slepton exchange [12] and in Minimal Supersymmetric Standard Model with right-handed neutrinos [13] through lepton number violating interactions accompanied with neutrinos). We consider magnitude of  $B, C, D, X, Y$  due to NSI not higher than  $\alpha$  due to various experimental constraints [14]. In (9), ( \* ) is denoted for complex conjugation. In (10),  $G_F$  is the Fermi constant and  $n_e$  is the electron number density.

The mixing matrix  $U'$  can be found out as  $U' = R_{23} W$ . Here,  $W$  is the normalized eigenvectors of  $\Delta m_{31}^2 M / (2E)$  calculated through perturbative technique. We follow the technique adopted in [15] for Standard Model interactions. Let us consider the case where only  $\epsilon_{12}$  and  $\epsilon_{13}$  are present as NSI and where  $\sin \theta_{13}$  is small and of the order of  $\alpha$  or less.  $M$  can be written as  $M = M^{(0)} + M^{(1)} + M^{(2)}$  where  $M^i$  contains terms of the order of  $\alpha^i$ . Then we can write

$$\begin{aligned} M^{(0)} &= \frac{\Delta m_{31}^2}{2E} \text{diag}(A, 0, 1), \quad M^{(1)} = \frac{\Delta m_{31}^2}{2E} \begin{pmatrix} \alpha s_{12}^2 & b & a \\ b^* & \alpha c_{12}^2 & 0 \\ a^* & 0 & 0 \end{pmatrix}, \\ M^{(2)} &= \frac{\Delta m_{31}^2}{2E} \begin{pmatrix} s_{13}^2 & 0 & -e^{-i\delta} \alpha c_{13} s_{12}^2 s_{13} \\ 0 & 0 & -e^{-i\delta} \alpha c_{12} s_{12} s_{13} \\ -e^{i\delta} \alpha c_{13} s_{12}^2 s_{13} & -e^{i\delta} \alpha c_{12} s_{12} s_{13} & -s_{13}^2 \end{pmatrix} \end{aligned} \quad (11)$$

where

$$a = c_{23}Y + e^{-i\delta} c_{13}s_{13} + Xs_{23}, \quad b = c_{23}X + c_{12}c_{13}\alpha s_{12} - Ys_{23}. \quad (12)$$

The eigenvalues of  $H$  upto second order in  $\alpha$  are

$$\begin{aligned}\frac{m_1'^2}{2E} &\approx \frac{\Delta m_{31}^2}{2E} \left[ A + \alpha s_{12}^2 + s_{13}^2 + \frac{|b|^2}{A} + \frac{|a|^2}{(-1+A)} \right], \\ \frac{m_2'^2}{2E} &\approx \frac{\Delta m_{31}^2}{2E} \left[ \alpha c_{12}^2 - \frac{|b|^2}{A} \right], \quad \frac{m_3'^2}{2E} \approx \frac{\Delta m_{31}^2}{2E} \left[ 1 - s_{13}^2 + \frac{|a^*|^2}{(1-A)} \right]\end{aligned}\quad (13)$$

In the same way we can calculate the eigenvalues keeping NSI in 23 block. Using eqs. (4) and (13) and putting  $\epsilon_{12}$  and  $\epsilon_{13}$  to zero one obtains the earlier known magic baseline condition [5] in presence of only SMI. For small  $\sin \theta_{13} \leq \alpha$  this condition is given by

$$L = 2n\pi/(\sqrt{2}G_F n_e). \quad (14)$$

The probability  $P_{\nu_e \rightarrow \nu_\mu}$  of oscillation expression after using the baseline condition in (4) for such small  $\sin \theta_{13}$  is (upto order  $\alpha^2$ )

$$P(\nu_e \rightarrow \nu_\mu) \approx 4 \frac{s_{23}^2}{(1-A)^2} |a|^2 \sin^2 \frac{\Delta m_{31}^2 (1-A)L}{4E} \quad (15)$$

One can see from this probability expression that  $|a|^2$  does not contain CP violating phase  $\delta$  when we consider only SMI but it does contain  $\delta$  when we keep NSI terms  $X$  and  $Y$  in  $a$ . This means that the magic baseline condition (14) is not valid when there is  $\epsilon_{12}$  and  $\epsilon_{13}$  as NSI. For brevity, we are not showing the detailed calculation of obtaining the probability expression if  $\epsilon_{22}$ ,  $\epsilon_{23}$  and  $\epsilon_{33}$  is considered. However, one may note that in such cases probability  $P(\nu_e \rightarrow \nu_\mu)$  upto order  $\alpha^2$  is same with that in presence of only SMI. Any correction due to NSI is present in higher order of  $\alpha$  only. So for small  $\sin \theta_{13}$ , the magic baseline condition in (14) is valid when NSI is present only in 23 block of  $M$  in (9).

However, the above conclusions related to magic baseline condition change if we consider large  $\sin \theta_{13} > \alpha$ . Let us discuss the perturbative approach for large  $\theta_{13}$ . This was considered in [16] earlier for SMI only. Here we use it for both SMI and NSI particularly in the context of magic conditions. We consider NSI in 12 and 13 elements. Then  $M$  in (9) can be written as  $M = M^{(0)} + M^{(1)} + M^{(2)}$  where

$$\begin{aligned}M^{(0)} &= \frac{\Delta m_{31}^2}{2E} \begin{pmatrix} A + s_{13}^2 & 0 & e^{-i\delta} s_{13} c_{13} \\ 0 & 0 & 0 \\ e^{i\delta} s_{13} c_{13} & 0 & c_{13}^2 \end{pmatrix}, \\ M^{(1)} &= \frac{\Delta m_{31}^2}{2E} \begin{pmatrix} \alpha s_{12}^2 & b & a_1 \\ b^* & \alpha c_{12}^2 & -e^{-i\delta} \alpha c_{12} s_{12} s_{13} \\ a_1^* & -e^{i\delta} \alpha c_{12} s_{12} s_{13} & 0 \end{pmatrix},\end{aligned}$$

$$M^{(2)} = \frac{\Delta m_{31}^2}{2E} \begin{pmatrix} -\alpha s_{13}^2 s_{12}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha s_{13}^2 s_{12}^2 \end{pmatrix} \quad (16)$$

where

$$a_1 = c_{23}Y - e^{-i\delta} c_{13}s_{13}\alpha s_{12}^2 + X s_{23}. \quad (17)$$

The eigenvalues upto second order correction in  $\alpha$  are

$$\begin{aligned} \frac{m_1'^2}{2E} &\approx \frac{\Delta m_{31}^2}{2E} \left[ \frac{1}{2}(A+1-x) + \alpha s_{12}^2 - \alpha s_{13}^2 s_{12}^2 + \frac{2|b|^2}{A+1-x} - \frac{|a|^2}{x} \right], \\ \frac{m_2'^2}{2E} &\approx \frac{\Delta m_{31}^2}{2E} \left[ \alpha c_{12}^2 - \frac{2|b|^2}{A+1-x} - \frac{2(\alpha c_{12}s_{12}s_{13})^2}{A+1+x} \right], \\ \frac{m_3'^2}{2E} &\approx \frac{\Delta m_{31}^2}{2E} \left[ \frac{1}{2}(A+1+x) + \alpha s_{13}^2 s_{12}^2 + \frac{|a^*|^2}{x} + \frac{2(\alpha c_{12}s_{12}s_{13})^2}{A+1+x} \right] \end{aligned} \quad (18)$$

where

$$x = (1 + A^2 - 2A \cos 2\theta_{13})^{1/2}. \quad (19)$$

In the same way we can calculate the eigenvalues keeping NSI terms in 23 block also. Using (18) the condition (4) may be written for large  $\sin \theta_{13} > \alpha$  as

$$L = 8En\pi / \{\Delta m_{31}^2 (A+1-x)\} \quad (20)$$

Unlike baseline condition in (14) the condition in (20) depends on  $\theta_{13}$  (in  $x$ ) and energy and also this does not give  $\delta$  independent probability as discussed below. For such large  $\sin \theta_{13}$  the probability  $P_{\nu_e \rightarrow \nu_\mu}$  with baseline condition in (20) is (upto order  $\alpha^2$ )

$$P(\nu_e \rightarrow \nu_\mu) \approx -4Re[Z] \sin^2 \frac{\Delta m_{31}^2 Lx}{8E} - 2Im[Z] \sin 2 \frac{\Delta m_{31}^2 Lx}{8E} \quad (21)$$

where

$$\begin{aligned} Z &= \frac{s_{23}^2}{(1-\xi^2)} \left[ -\xi^2 k^2 - \frac{a_1 \xi^2 k^2}{x} + \frac{a_1^* \xi^3 k^2}{x} - \frac{|c_{23}Y + s_{23}X|^2 \xi^4 k^2}{x^2} + \frac{|c_{23}Y + s_{23}X|^2 \xi^2}{x^2} \right] \\ &+ \frac{c_{23}s_{23}}{(1-\xi^2)} \left[ -\frac{4(c_{23}\beta^* - s_{23}\gamma^*)\xi k}{(A+1+x)} \right] \end{aligned} \quad (22)$$

and

$$\begin{aligned} \beta &= C c_{23} + B s_{23}; \quad \gamma = D c_{23} + C s_{23}; \quad \xi = (-A + \cos 2\theta_{13} + x) \operatorname{cosec} 2\theta_{13}; \\ k &= 1/[1 + (-A + \cos 2\theta_{13} + x)^2 \operatorname{cosec}^2 2\theta_{13}]^{1/2} \end{aligned} \quad (23)$$

This probability is not independent of  $\delta$  due to the presence of  $a_1$  in  $Z$ . So it is not possible to get magic baseline condition (resulting in  $\delta$  independent probability  $P_{\nu_e \rightarrow \nu_\mu}$  upto order  $\alpha^2$ ) for large  $\sin \theta_{13} > \alpha$  with or without NSI in any elements in  $M$ .

However, if we consider some magic condition on neutrino energy then it is possible to get  $\delta$  independent probability  $P_{\nu_e \rightarrow \nu_\mu}$  for both small and large  $\theta_{13}$  and also with and without NSI. Using condition (6) and considering  $P_{\nu_e \rightarrow \nu_\mu}$  upto order  $\alpha^2$  the magic energy condition for small  $\sin \theta_{13} \leq \alpha$  is written as

$$E = \Delta m_{31}^2 / \left( \pm 4n\pi/L + 2\sqrt{2}G_F n_e \right). \quad (24)$$

Using the above energy condition for such small  $\sin \theta_{13}$  with NSI terms (upto order  $\alpha^2$ )

$$P(\nu_e \rightarrow \nu_\mu) \approx \frac{4c_{23}^2}{A^2} |c_{23}X + c_{12}c_{13}\alpha s_{12} - Y s_{23}|^2 \sin^2 \left( \frac{\Delta m_{31}^2 AL}{4E} \right). \quad (25)$$

With  $X = Y = 0$  this corresponds to Standard Model result. Unlike (15) this is independent of  $\delta$  even with NSI terms. This is one important advantage of using magic energy condition instead of magic baseline condition even for  $\sin \theta_{13} \leq \alpha$ . However, as the condition is on energy it might be useful to consider monoenergetic neutrino beam as source [10, 11] to study such  $\delta$  independent probability. The other alternative way to study such probability might be to consider very small energy bins for neutrino energy which satisfies approximately the above energy condition in (24).

Using condition (6) and considering  $P_{\nu_e \rightarrow \nu_\mu}$  upto order  $\alpha^2$ , the magic energy condition for large  $\sin \theta_{13} > \alpha$  is written as

$$E = \Delta m_{31}^2 S \cos 2\theta_{13} L^2 \left( -1 \pm \sqrt{1 + Q/R} \right) / (2Q) \quad (26)$$

where

$$Q = (2n\pi)^2 - S^2 L^2; R = S^2 L^2 \cos^2 2\theta_{13}; S = \sqrt{2}G_F n_e.$$

Although this condition depends on  $\theta_{13}$  but with presently allowed values of  $\theta_{13}$  [2] this dependence is not so significant as shown later in Figure 1 in which  $E$  satisfying condition (24) overlaps on  $E$  satisfying condition (26). It is important to note here that unlike magic baseline condition (20,) this energy condition (26) results in  $\delta$  independent probability as shown below. Using condition (26) and considering NSI terms and  $\sin \theta_{13} > \alpha$ , the probability  $P_{\nu_e \rightarrow \nu_\mu}$  is written as (upto order  $\alpha^2$ )

$$P(\nu_e \rightarrow \nu_\mu) \approx \frac{16|b|^2 \xi^2 c_{23}^2}{(1 + \xi^2)(A + 1 - x)^2} \sin^2 \left( \frac{\Delta m_{31}^2 L(A + 1 - x)}{8E} \right). \quad (27)$$

Unlike the energy condition (26), the probability of oscillation depends significantly on  $\theta_{13}$ . This probability is  $\delta$  independent with or without NSI.

Due to present ambiguity in the sign of  $\Delta m_{31}^2$ , with (+) sign to  $\Delta m_{31}^2$  for normal and with (-) sign for inverted hierarchy of neutrino masses, the energy conditions will be different. Apart from hierarchy sign there is further consideration of choosing signs in energy conditions as seen in (24) and (26). The requirement of positive energy allows considering both of those signs in the energy conditions for normal hierarchy and considering only (-) sign for inverted hierarchy. For normal hierarchy for (-) sign in the conditions,  $L > \frac{\sqrt{2}n\pi}{G_F n_e}$  but there is no bound for (+) sign. For inverted hierarchy  $L < \frac{\sqrt{2}n\pi}{G_F n_e}$ . Due to singularity at  $L = \frac{\sqrt{2}n\pi}{G_F n_e}$ , neutrino energy  $E$  satisfying energy conditions are not possible at magic baseline length.

Using magic energy condition one could resolve ambiguities in  $\delta - \theta_{13}$  (which usually happens for non-magic neutrino energy) as the probability is  $\delta$  independent. Furthermore, using energy condition one may also try to find the neutrino mass hierarchy. For illustration, let us consider say nature admits normal hierarchy and  $\sin \theta_{13} \leq \alpha$  and for simplicity say NSI is absent. Certain neutrino energy has been fixed by energy condition in (24) with appropriate choice of  $n$  value for which monoenergetic neutrino will be feasible in experiment. Under such conditions the probability in (25) is independent of  $\theta_{13}$  upto order  $\alpha^2$ . So the probability has fixed value and normal hierarchy could be verified by experiment upto order  $\alpha^2$  from the number of  $\mu$  events observed at the detector. This number in general, differs from that which one could have obtained for inverted hierarchy in this case. The reason is that, for inverted hierarchy the same neutrino energy will not correspond to magic energy anymore. In fact, in this case, one can show that the difference in probability of oscillation  $P_{\nu_e \rightarrow \nu_\mu}^{N(magic)}$  for normal hierarchy with magic energy condition with that for inverted hierarchy  $P_{\nu_e \rightarrow \nu_\mu}^{I(non-magic)}$  without any magic energy condition is

$$\begin{aligned} P_{\nu_e \rightarrow \nu_\mu}^{N(magic)} - P_{\nu_e \rightarrow \nu_\mu}^{I(non-magic)} &= -\frac{1}{(1+A)^2} \sin^2 2\theta_{13} \sin^2 \theta_{23} \sin^2 \left( \frac{(1+A)\Delta m_{31}^2 L}{4E} \right) \\ &\quad + \frac{1}{A(1+A)} \alpha \sin 2\theta_{13} \cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \times \\ &\quad \cos \left( \delta - \frac{\Delta m_{31}^2 L}{4E} \right) \sin \left( \frac{A\Delta m_{31}^2 L}{4E} \right) \sin \left( \frac{(1+A)\Delta m_{31}^2 L}{4E} \right) \end{aligned}$$

which does not vanish in general, for any value of  $\delta$  unless  $\sin \theta_{13}$  vanishes. However, after fixing  $L$  and  $E$  for the experiment one may check whether this difference vanishes or not. So for normal hierarchy, in general, one is supposed to get different number of  $\mu$  events at the detector than that for inverted hierarchy and neutrino mass hierarchy may be resolved for  $0 < \sin \theta_{13} \leq \alpha$ . If the number of  $\mu$  events does not match with the expected one for

normal hierarchy at magic energy, one may try the magic energy for inverted hierarchy in the experiment. Similar to the above expression, one can show that  $P_{\nu_e \rightarrow \nu_\mu}^{I(magic)} - P_{\nu_e \rightarrow \nu_\mu}^{N(non-magic)}$  will not vanish in general, for any value of  $\delta$  unless  $\sin \theta_{13}$  vanishes.

For larger  $\sin \theta_{13} > \alpha$ , the above difference has more complicated form. Same method can be adopted in this case also to resolve hierarchy, provided that the difference of  $P_{\nu_e \rightarrow \nu_\mu}^N$  (for normal hierarchy) and  $P_{\nu_e \rightarrow \nu_\mu}^I$  (for inverted hierarchy) does not vanish for  $L$  and  $E$  value considered in the experiment (where  $E$  could be magic energy for either normal or inverted hierarchy). In case, it vanishes either for particular combination of  $\sin \theta_{13}$  or  $\delta$  one may consider for the same baseline, a different neutrino magic energy by changing  $n$  value in the magic energy condition for which such difference may not vanish.

We now, illustrate the use of such magic energy conditions to get  $\delta$  independent probability and discuss the experimental feasibility. One is required to fine-tune the energy of the monoenergetic neutrino beam. In the electron capture facility as discussed at the beginning, the neutrino energy may be fixed by appropriately choosing the boost of the ion source. We consider the monoenergetic neutrino beam for the ion type  $^{150}Dy$  with neutrino energy  $E_r$  at rest given by 1.4 MeV as suggested in ref. [10]. We have chosen Lorentz boost  $\gamma$  in the range of 90 -195 such that the magic energy condition is satisfied. The neutrino energy  $E$  is fixed in the forward direction by the boost as  $E = E_r \gamma$ . We have assumed flux of  $10^{18}$  neutrinos per year. We are considering a baseline of length 650 Km from CERN to megaton water Cerenkov detector possibly located at Canfranc in Spain. For such baseline the constant matter density has been approximated to be 4.21 gm/cc. For our subsequent sensitivity analysis of oscillation parameters we mention here the detector characteristics also for such experimental set-up. [7]: (a) Fiducial mass = 500 Kton (b) Detection efficiency ( $\epsilon$ ) = 50 % (c) Charge identification efficiency ( $I_e$ ) = 95 %.

In Figure-1 we have shown the energy versus length of baseline satisfying magic energy conditions. Condition (24) has been considered for any  $\sin \theta_{13} \leq \alpha$  and condition in (26) for  $\theta_{13} = 5, 8, \text{ and } 12$  degrees. However, it is seen from the figure that the plots with different  $\theta_{13}$  are almost overlapping with each other indicating very small change in energy  $E$  with  $L$  due to variations of unknown parameter  $\theta_{13}$ . In plotting Figure 1, instead of  $n = 1$  we have considered  $n = 2$  in the energy conditions (24) and (26) so that for the above-mentioned baseline of length 650 Km, the magic neutrino energy lies in experimentally feasible range.

Finally we discuss the sensitivity in measuring the unknown oscillation parameters like  $\theta_{13}$  and  $\Delta m_{31}^2$  in the experimental set-up with monoenergetic neutrino beam [10, 11]. The

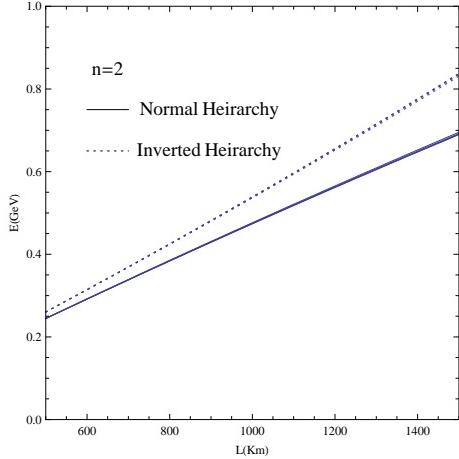


Figure 1: Plots for neutrino energy ( $E$ ) versus length ( $L$ ) of baseline using energy conditions (24) for any  $\sin\theta_{13} \leq \alpha$  and (26) for  $\theta_{13} = 5, 8, 12$  for  $n = 2$  for both hierarchies.

number of  $\mu$  events expected at the detector due to  $\nu_e \rightarrow \nu_\mu$  oscillation is given by

$$N_\mu = T n_n I_e \epsilon E \phi(E) \sigma_{\nu_\mu}(E) P_{\nu_e \rightarrow \nu_\mu}(E) \quad (28)$$

where  $T$  = time period ,  $n_n$  = number of target nucleons,  $\phi(E)$  = flux,  $\sigma_{\nu_\mu}(E)$  = detection cross-section. As we are considering monoenergetic neutrino beam so the energy resolution function on which normally number of events depends, may be considered to be effectively 1 and as such is not mentioned in eq. (28). In figure 2 we have shown for both hierarchies the variation of the number of events expected for a priod of 5 years with  $\theta_{13}$  in absence of any NSI and with NSI respectively. The number of  $\mu$  events are quite large as neutrinos have some fixed energy instead of Gaussian distribution of energy. Number of events with inverted hierarchy is found to be slightly higher than that for normal hierarchy for different values of  $\theta_{13}$  with or without NSI.

In plotting figures 4 and 5 we define  $\chi^2_{total}$  as

$$\chi^2_{total} = \left( \frac{N^{expt} - N^{th}}{\Delta N} \right)^2 + \left( \frac{|\Delta m_{31}^2| - |\Delta m_{31}^2(true)|}{\sigma(\Delta m_{31}^2)} \right)^2 + \left( \frac{\sin^2 2\theta_{23} - \sin^2 2\theta_{23}(true)}{\sigma(\sin^2 2\theta_{23})} \right)^2 \quad (29)$$

where  $N^{expt}$  and  $N^{th}$  stands for experimental and theoretical number of  $\mu$  events respectively at the detectors and the error in  $N^{th} = \Delta N = \Delta N_{pert} + \Delta N_{\alpha^2}$ . Here,  $\Delta N_{pert}$  and  $\Delta N_{\alpha^2}$  are the differential change in  $N^{th}$  considering perturbative expression of  $P_{\nu_e \rightarrow \nu_\mu}$  in (27) and the perturbative error of order  $\alpha^2$  in (27) respectively. The  $\Delta N_{\alpha^2}$  takes care of maximum possible correlated error due to any true value of  $\delta$  and we have assumed democratic

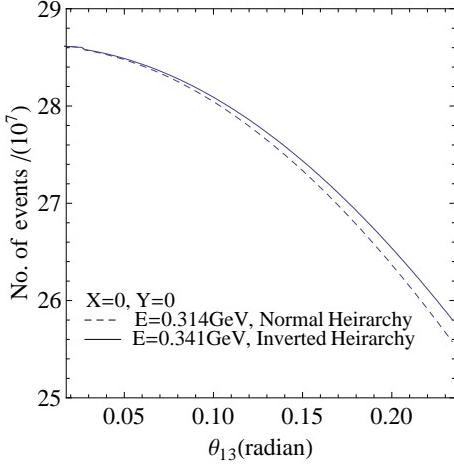


Figure 2: Number of  $\mu$  events versus mixing angle  $\theta_{13}$  with 5 years running without NSI (i.e  $X = Y = 0$ ).

form of correlation matrix. Unlike magic baseline condition for magic energy condition the perturbative expression of probability depends on  $\cos^2 \theta_{13}$  instead of  $\sin^2 2\theta_{13}$  and thus resulting in large number of events ( $\sim 10^7$ ) even for small  $\theta_{13}$ . So unlike [7] instead of Poisson distribution, we have considered Gaussian distribution for error in  $N$ . In evaluating 2nd and 3rd terms on right side of eq.(29) we have considered  $\Delta m_{31}^2(\text{true}) = 2.5 \times 10^{-3} \text{ eV}^2$ ,  $\sin^2 2\theta_{23}(\text{true}) = 1.0$ ,  $\sigma(\Delta m_{31}^2) = 1.5\%$ ,  $\sigma(\sin^2 2\theta_{23}) = 1\%$ . Based on various experimental data set [7] the following  $3\sigma$  constraints on the following oscillation parameters have been considered :

$$2 \times 10^{-3} \text{ eV}^2 < |\Delta m_{31}^2| < 3.2 \times 10^{-3} \text{ eV}^2;$$

$$35.67^\circ < \theta_{23} < 55.55^\circ.$$

In figure 4 and 5 we have shown the precision in measuring  $\cos^2 \theta_{13}$  which appears in the expression of  $P_{\nu_e \rightarrow \nu_\mu}$  satisfying magic energy condition. This is defined by

$$\text{Precision} = \frac{\cos^2 \theta_{13}(\text{min}) - \cos^2 \theta_{13}(\text{max})}{\cos^2 \theta_{13}(\text{min}) + \cos^2 \theta_{13}(\text{max})} 100\% \quad (30)$$

in which  $\cos^2 \theta_{13}(\text{min})$  and  $\cos^2 \theta_{13}(\text{max})$  are the smallest and largest values respectively of  $\cos^2 \theta_{13}$  at the given confidence level. We have shown the precision at  $1\sigma$  and  $3\sigma$  level with and without NSI for both the hierarchies as mentioned in the figures. In finding precision in measurement of  $\cos^2 \theta_{13}$  in presence of NSI, we have assumed that the strength of NSI couplings are known from some other experiments. The number of muon events at the

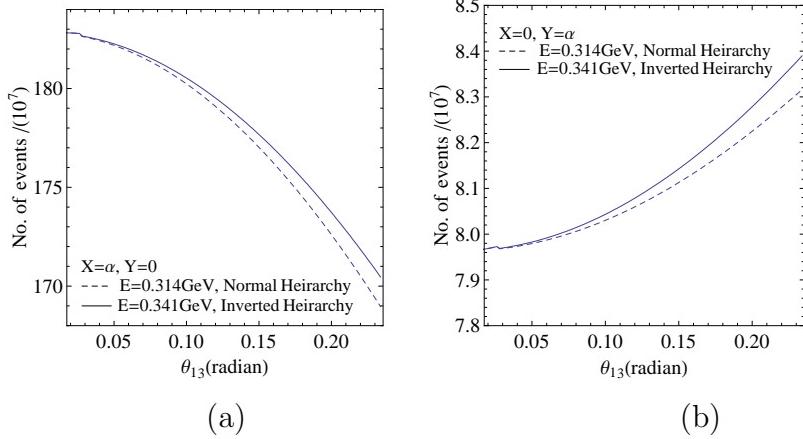


Figure 3: Number of  $\mu$  events versus mixing angle  $\theta_{13}$  with 5 years running with NSI  $X$  and  $Y$ .

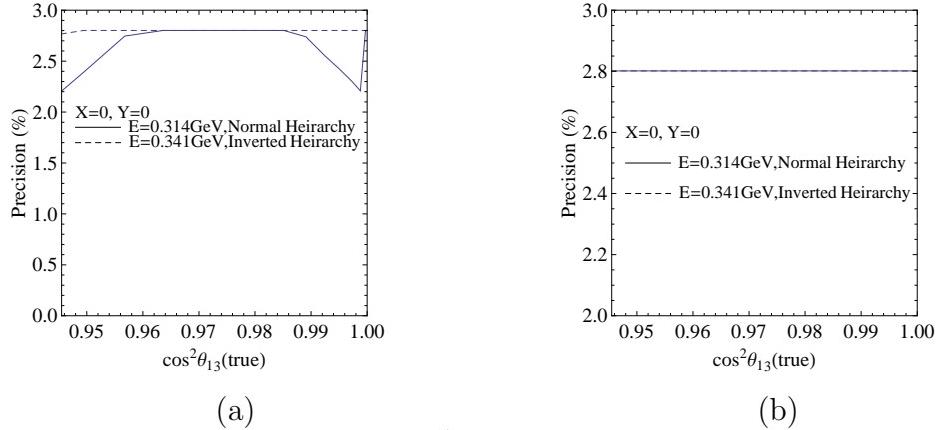


Figure 4: Precision in the measurement of  $\cos^2 \theta_{13}$  without NSI expected with 5 years running at  $1\sigma$  in (a) and at  $3\sigma$  in (b).

detector changes when such NSI are included and thus changes the level of precision in measurement of  $\cos^2 \theta_{13}$ . The precision as defined above is better for its lower values. In general in both figures 4 & 5 the precision initially deteriorates with increase of  $\cos^2 \theta_{13}$  values and after reaching some limiting values of  $\cos^2 \theta_{13}$  the precision starts improving and finally for larger values of  $\cos^2 \theta_{13}$  again it deteriorates. However, in some cases the change in precision is insignificant with the variations of  $\cos^2 \theta_{13}$  as seen in figures 4(b) and 5 (c) & (d). From figures the differences in precision for normal and inverted hierarchies are particularly found at  $1\sigma$  level except in figure 5 (c) where non-zero value of  $Y$  as NSI has been considered. Comparing three cases - (a) no NSI (b) NSI with only non-zero  $X$  (c) NSI with only non-zero  $Y$ , it is seen from figures 4 & 5 that the precision in measurement of  $\cos^2 \theta_{13}$  is best for (b)

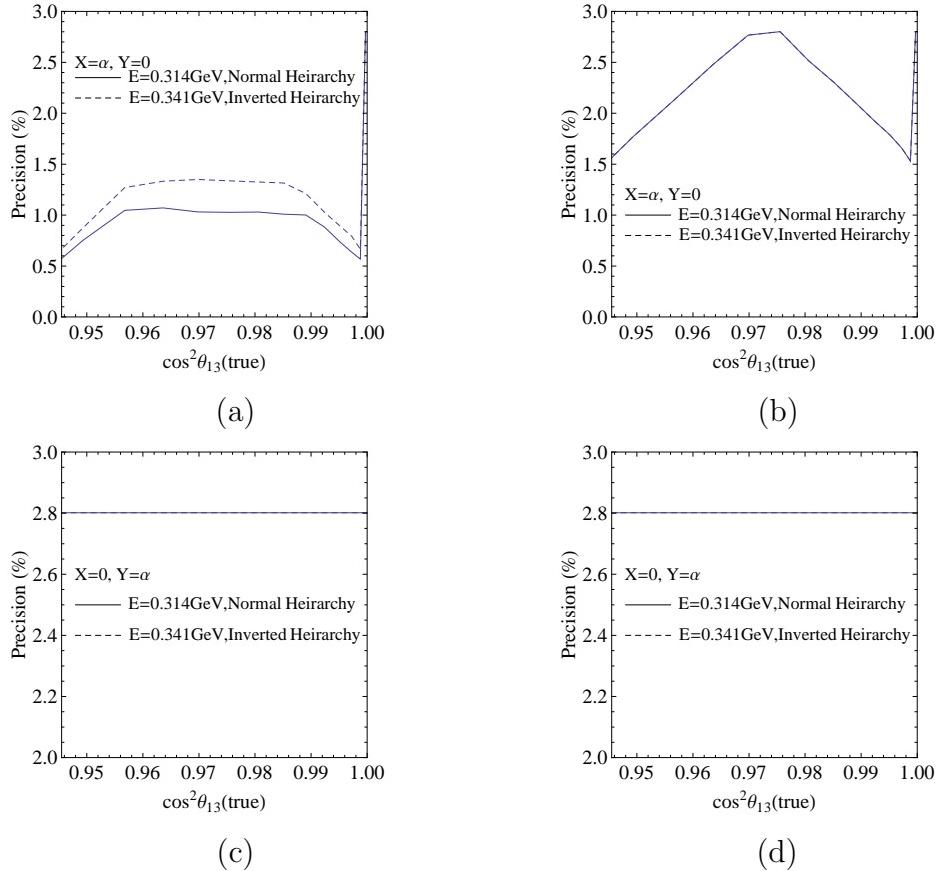


Figure 5: Precision in the measurement of  $\cos^2 \theta_{13}$  with NSI expected with 5 years running at  $1\sigma$  in (a) and (c) and at  $3\sigma$  in (b) and (d).

and the precision is better for (a) than that for (c). Actually more the number of  $\mu$  events the better is the precision as these cases may be seen from figures 2 & 3.

As concluding remarks we mention that to get  $P_{\nu_e \rightarrow \nu_\mu}$  almost independent of unknown  $CP$  violating phase  $\delta$  one may consider either magic baseline condition on the length of baseline or the magic neutrino energy condition. However, there are some disadvantages in considering the magic baseline condition which are not present when magic energy condition is considered. The magic baseline condition exists only for small  $\sin \theta_{13} \leq \alpha$ . Also this condition exists when NSI are considered in only 23 block of effective neutrino mass matrix  $M$  (as  $P_{\nu_e \rightarrow \nu_\mu}$  upto order  $\alpha^2$  is independent of NSI in 23 block). However, magic baseline condition is not possible if NSI terms are present in 12 , 13 elements of  $M$ . For large  $\sin \theta_{13} > \alpha$  using magic baseline condition in (20) it is not possible to get  $\delta$  independent probability  $P_{\nu_e \rightarrow \nu_\mu}$  with or without NSI. Magic baseline condition will also depend on neutrino

energy for  $\sin \theta_{13} > \alpha$ . Also to place neutrino detector at a location satisfying magic baseline condition may not be always feasible. Some of the drawbacks mentioned above in considering magic baseline condition may be overcome by considering the condition on neutrino energy. Under magic neutrino energy condition for both small and large  $\sin \theta_{13}$  and also with or without NSI one gets  $\delta$  independent probability  $P_{\nu_e \rightarrow \nu_\mu}$ .

Using energy condition there is scope to find out the hierarchy of neutrino masses and to obtain overall good precision in the measurement of  $\cos^2 \theta_{13}$  over the allowed range of  $\theta_{13}$  as discussed earlier. Depending on the presence or absence of NSI, the number of  $\mu$  events could significantly differ as shown in Fig. 2 & 3 and could signal the presence of new physics. If the experimental data indicates the presence of NSI then to find NSI as well as  $\theta_{13}$  one may consider changing the magic neutrino energy by changing the Lorentz boost in the same experimental set-up. Then NSI terms like  $\epsilon_{12}$  and  $\epsilon_{13}$  as well as  $\theta_{13}$  may be known from  $P_{\nu_e \rightarrow \nu_\mu}$  in (25) for  $\sin \theta_{13} < \alpha$  or (27) for  $\sin \theta_{13} > \alpha$  after matching those probabilities with experimental data on number of  $\mu$  events using eq. (28). In long baseline experiments, monoenergetic neutrino beam as source with neutrino energy satisfying magic energy condition, could be highly useful in future advanced precision measurements of neutrino oscillation parameters, in resolving hierarchies of neutrino masses and also in searching NSI of neutrinos with matter.

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